Corporate Management Review
Vol. 36 No. 1, 2016
pp. 97-123

Simplifying the valuation of reverse annuity mortgages
不動產逆向抵押貸款年金的簡化評價方法

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Abstract: This research derives an approximate pricing formula for valuing reverse annuity mortgages that allows house prices and interest rates to be stochastic. Our approximation approach reduces computational intensity, because it only requires an expectation (average) and a variance of the termination time. We compare the results from our approximate pricing formula with results from simulations and find that the formula provides a close approximation to the simulation results. We conclude that these approximating formulae are useful in valuing and hedging reverse mortgage portfolios, whereas simulations are computationally prohibitive. We further note that the difference between the results of the approximation formula and the simulation is small and generally less than 1%.

Keywords: Reverse mortgage, Annuity, Option pricing

1. Introduction

Financial engineers are relentless in designing new products that can assist in solving important societal financial problems. Because the societal issues that these financial instruments aim to solve are important, their development often leads to them to becoming very popular and significantly improving the quality of people’s lives. For example, various financial products allow for the reduction

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of risk or for people to choose preferred payment patterns for housing mortgages. Unfortunately, due to their complexity, valuing many of these financial instruments is a significant challenge for the financial industry.

One such financial instrument has been referred to as a reverse mortgage, reverse annuity mortgage, or a home equity conversion mortgage. The motivation for creating reverse mortgages was that many seniors have built substantial equity in their homes, but lack the liquid financial resources needed to enjoy their golden years. Moreover, for many of these homeowners, the equity in their homes is their largest financial asset.

Reverse mortgages were designed to address the problem of senior homeowners having a significant portion of their wealth tied up in an illiquid home by allowing them to obtain a loan by pledging their home as collateral. These loans permit the homeowner to convert the home equity into either a lump sum payment or multiple monthly payments. The homeowner’s obligation to repay the loan is then deferred until the owner dies or sells the home. Thus, a reverse mortgage is the opposite of a conventional mortgage in that the borrower receives payments from the lender in lieu of making payments to the lender.

Although reverse mortgages are only a small part of the total mortgage market, their popularity has increased substantially in recent years and is expected to continue to grow.

This is due, in part, to the 1989 Home Equity Conversion Mortgage (HECM) program insured by the U.S. Federal Housing Administration (FHA). Under the HECM program, reverse mortgages are insured by the FHA and allow homeowners that are at least 62 years old to withdraw equity from their homes in the form of a lump sum, monthly payments, a line of credit, or any combination of the above. These reverse mortgages are typically non-recourse loans and the lenders can only claim repayment of principle and interest from the proceeds of selling the collateralized home. If the value of the home is insufficient to cover the outstanding balance of the reverse mortgage, the lender will suffer a loss.

Because reverse mortgages permit borrowers to stay in their home until they die or move, the time until mortgage termination is uncertain. While the central premise of valuation is to discount expected cash flows at the required rate of return, doing this in practice for reverse mortgages is a challenge due to the undefined termination time.

Despite the importance of these financial instruments, little research has addressed reverse mortgage valuation. Only Tsay *et al*. (2014) examined this topic and only for mortgages with a lump sum payment. We fill the gap in this literature by developing an approximate pricing formula for reverse mortgages with an annuity payment and compare the results of our model to those of Monte Carlo simulation. Because mortgage termination time is a critical issue, we examine the robustness of our results using an exponential distribution and life table to estimate termination time.

It is worth mentioning that our research assumed that investors are rational. But in fact, Chang *et al*. (2012) proposed that herd behavior of investors would affect the price and volatility of assets. Risk-neutral pricing method is used in asset pricing in our research, we converse the distribution of cash flow into risk-adjusted cash flow by risk measure and use risk free rate to discount cash flow. We use this method because many researches mentioned about asset's reasonable required rate of return. For example, Chen (2011) mentioned that a bank's off-balance financing will affect investor's required rate of return and then further affect valuation of Bank's loan securitizations. Chi (2013) and Chi (2015) pointed out that bankruptcy or reorganization of borrowers may affect banks and cause investor's required rate of return to change. Also, Yeh and Yu (2014) pointed out that accounting system will affect asset pricing, and that fair value accounting is more capable of pricing assets value. All these illustrate that there too many factors to consider when calculating reasonable require rate of return of bank assets, thus, we can converse the probability measure by Girsanov theorem to avoid those problems.

The remainder of the paper is organized as follows. The next section provides the development of our approximation formula. Section 3 presents a comparative analysis of testing our approximation formula against results from Monte Carlo simulation. Section 4 concludes.

2. Methodology

Reverse mortgages generally may take one of two forms: term or tenure. Under the term form of a reverse mortgage, the borrower is provided with income for a number of specified periods. Under the tenure form of a reverse mortgage, the borrower is provided with income for as long as he/she continues to stay on the property. Thus, in most cases, reverse mortgages are similar to European option contracts that bear no prepayment risk. In other words, this similarity occurs, because the borrowers' repayment of principal is deferred until death or the sale of the property. Likewise, a European option can only be exercised at maturity. Consequently, like options, reverse mortgage loan contracts can be valued by an approximate solution. In this section we develop an approximation solution under a Gaussian framework.

2.1 Fundamental structure

We attempt to value the payoff that takes place for a tenure form of reverse mortgage at the time of death or sale of the house:

$$V_{1,T} = min\{H_T, B_T\} = B_T - \max\{0, B_T - H_T\}, \tag{1}$$

where H_t is the house price at time T, and B_T is the balance of the reverse mortgage at time T. We denote the reverse mortgage value (payoff) at time T as $V_{1,t}$. Note that T is the termination time of the reverse mortgage and is assumed to be either the death of the mortgagee or the sale of the house. Because Boehm and Ehrhardt (1994) and Klein and Sirmans (1994) report that the typical reverse mortgage borrower is 75 years of age, these two events may occur at nearly the same time.

Our objective is to find the value of a reverse mortgage, $V_{1,t}$, at any time t. We assume that the balance of the reverse mortgage at time T is:

$$\mathrm{B_T} = \frac{\mathrm{Xe^{r_cT}}(1-\mathrm{e}^{-r_cT})}{1-\mathrm{e^{-r_c\Delta}}} = X\,\frac{e^{r_cT}-1}{1-e^{-r_c\Delta}} = X\ f_T \tag{2}$$

$$\mathrm{B_T} = B_t e^{r_c(T-t)}\,\frac{1-e^{-r_cT}}{1-e^{-r_ct}}, \tag{2-1}$$

where r_c represents the fixed contract rate, X is the monthly payment, and B_0 is the initial amount borrowed. The length of each time period is Δ, f_T is the final value separating T into T/Δ and receiving \$1 per period, X is the amount of annuity received every period, and B_T is the final value of annuity after the bank paysan annuity to mortgage borrowers for T years. In our research, Δ=1/12, and T is a multiple of 1/12. The value of the reverse mortgage at any time t before T is:

$$V_{1,t} = E_t\left\{e^{-r(T-t)}\min[H_T, B_T]\right\}, \tag{3}$$

where *r* represents the static risk-free rate.

2.2 Stochastic house prices

If the house price is stochastic, then we can write the reverse mortgage value at time *T* as:

$$V_{2,T} = min\left\{\widetilde{H_T}, B_T\right\}). \tag{4}$$

We assume the house price, *H*, follows a Geometric Brownian motion process:

$$dH = \mu_h H dt + \sigma_h H dw_h, \tag{5}$$

whereμ_h is the expected growth rate of the house price, σ_h is the volatility of the house price growth rate, and $w_h$is a Wiener process of the house price growth rate. We define: $dW_h^Q = \frac{\mu_H - r_t}{\sigma_H}dt + dW_h$, then $dH = r_t H dt + \sigma_h H dW_h^{\,Q}$, where r_t is the risk-free rate. Girsanov's theorem states that there exists a risk-neutral probability measure Q under which W_h^Q is a Brownian motion process with a risk probability measure Q: $V_{1,t} = E_t^Q\{e^{-\int_t^T r_s ds} \min(H_T, B_T)\}$.

In order to value a reverse mortgage, it can be viewed as either a portfolio consisting of buying a zero coupon bond and selling a put option with an exercise price equal to the house price at B_T, or a portfolio consisting of buying a house and selling acall option with exercise house price at B_T. If the value of the collateralized house does not exceed the outstanding loan balance, then the lender can only receive the proceeds from the sale of the residential property. As a result, upon the death of the homeowner the lender will receive the following payoff:

$$V_{2,T} = min[H_T, B_T] = B_T + min[H_T - B_T, 0] = B_T - max[B_T - H_T, 0] \quad (6)$$

or

$$V_{2,T} = min[H_T, B_T] = H_T + min[0, B_T - H_T] = H_T - max[0, H_T - B_T]. \quad (7)$$

Corollary 1: Given that the house price, H_t, is governed by a Geometric Brownian motion process and that the interest rate r and termination time T of a reverse mortgage are deterministic, the value of a reverse mortgage at time t before T is:

$$V_{2,t} = B_t - PUT_t = B_T e^{-r(T-t)} - B_T e^{-r(T-t)} N(-d_2) + H_t N(-d_1), \quad (8)$$

or

$$V_{2,t} = H_t - H_t N(d_1) + X f_T e^{-r(T-t)} N(d_2), \quad (9)$$

where

$$d_2 = -\frac{ln\frac{Xf_T}{H_t} - (r - \frac{\sigma_h^2}{2})(T-t)}{\sigma_h\sqrt{T-t}}$$

$$d_1 = d_2 + \sigma_h\sqrt{T-t}.$$

2.3 Stochastic interest rates

The empirical results of Meen (2000) and Jud and Winkler (2002) point out that there is as ignificantly negative relationship between interest rate sand house prices. Summers (1981) shows that low inflation and low interest rates favoractivity in the stock market and bond market, but depress the real estate market. However, Harris (1989) asserts that the rise in nominal interest rates can cause inflation, which results in an expected increase in the housing prices.

There currently exists no consensus about the relationship between interest rates and housing prices. Some studies consider the relationship to be positive (Harris, 1989; Peiser and Smith, 1985; Summers, 1981), while others consider it to be negative (Kau and Keenan,1981; Reichert,1990). Despite these conflicting conclusions among researchers, it is undeniable that interest rates and housing prices are related. At the beginning stage of recovery of the real estate market, short-term housing supply is limited. Under the expectation of a growing real estate market, investment demand will start to increase. House prices then begin to rise due to greater demand and low supply in the market.

We continue to assume that the house price, H, follows a Geometric Brownian motion process and also assume that the interest rate follows the Ornstein-Uhlenbeck (OU) process:

$$dr_t = \alpha(\mu_r - r_t)dr + \sigma_r dw_r, \tag{10}$$

where μ_r is the mean reversion level, which is the weighted average level of the interest rate process, and α is the mean reversion speed that indicates the strength of the "attraction" described in μ_r.[2] For small values of α, this effect disappears. For large values of α, r_t converges quickly to μ_r. Here, σ_r is the volatility of the interest rate, and w_r is a Wiener process of the interest rate.

Corollary 1 is a special case of Corollary 2 when we set the volatility of interest rate, σ_r, equal to zero and mean reversion speed, μ_r, equal to r (which is deterministic). In this setting, the increment of r_t, dr_t, will be zero, and the interest rate is equal to r_0 at any time *t*.

Corollary 2: Given that the house price, H_t, is governed by a Geometric Brownian motion process and the interest rate r follows the Ornstein-Uhlenbeck (OU) process, if the termination time of reverse mortgage T is deterministic, then the reverse mortgage value at time t before T is:

$$V_{3,t} = H_t N\left(\frac{k-\sigma_h\sqrt{\tau}}{\sqrt{1+n_0^2(1-\rho^2)S_r}}\right) + X f_\tau e^{-M_r+\frac{S_r}{2}} N\left(\frac{-k+\rho\sqrt{S_r}}{\sqrt{1+n_0^2(1-\rho^2)S_r}}\right), \tag{11}$$

Where

$$k = \frac{(m - nM_r)}{1+n\rho\sqrt{S_r}};\ m = \frac{ln\frac{Xf_t}{H_t}+\frac{1}{2}\sigma_h^2\tau}{\sigma\sqrt{\tau}},$$

[2] The reason why we use the OU process is that it permits mean-reverting interest rates. We find that sigma_r has little effect on asset pricing during simulation.

$$\text{n}=\frac{1}{\sigma\sqrt{\tau}}.,\ n_0=\frac{n}{1+n\rho\sqrt{S_r}},\ \text{and}\sigma_\text{b}^2=(1-\rho^2)S_{r.}$$

$$M_r=\mu_r(T-t)+(r_t-\mu_t)\frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$S_r=\frac{\sigma_r^2}{\alpha^2}\left[(T-t)-\frac{1-e^{-\alpha(T-t)}}{\alpha}-\frac{1}{2}\alpha\left(\frac{1-e^{-\alpha(T-t)}}{\alpha}\right)^2\right].$$

Proof:

Set $\mathrm{R}(\tau)=\int_\mathrm{t}^\mathrm{T}\mathrm{r_s ds}$.Based on Equations (5) and (10), we obtain Equations (12) and (13) by using Ito's Lemma.

$$\mathrm{R}(\tau)\sim\mathrm{N}\left(\mu\tau+(r_t-\mu)\frac{1-e^{-\alpha\tau}}{\alpha}\ ,\ \frac{\sigma_r^2}{\alpha^2}\left(\tau-\frac{1-e^{-\alpha\tau}}{\alpha}-\frac{1}{2}\alpha\left(\frac{1-e^{-\alpha\tau}}{\alpha}\right)^2\right)\right)=\mathrm{N}(M_r S_r). \quad (12)$$

$$H_T=H_t e^{\int_t^T r_s ds-\frac{1}{2}\sigma_h^2\tau+\sigma_h\sqrt{\tau}z}. \quad (13)$$

Here, z follows a standard normal distribution and $\tau=T-t$.

We first can derive the value of a reverse mortgage as:

$$V_{3,t}=E\left(e^{-R}\max\left(\mathrm{H_T}-B_T,0\right)\right). \quad (14)$$

$H_T<X_T$ implies:

$$\mathrm{H_T}=\mathrm{H_t}e^{R-\frac{\sigma_h^2\tau}{2}+\sigma\sqrt{\tau}z}<Xf_\tau \quad (15)$$

$$\mathrm{Z}\le\frac{\ln\frac{Xf_t}{H_t}-\left(R-\frac{1}{2}\sigma_h^2\tau\right)}{\sigma_h\sqrt{\tau}}=\mathrm{m}-\mathrm{nR}. \quad (16)$$

Here, $\mathrm{m}=\frac{ln\frac{Xf_t}{H_t}+\frac{1}{2}\sigma_h^2\tau}{\sigma_h\sqrt{\tau}}$, $\mathrm{n}=\frac{1}{\sigma_h\sqrt{\tau}}$.

After finding the reasonable range, we can derive the value of a reverse mortgage as Equation 17, which is the joint probability density function of (z,R):

$$\mathrm{f(z,R)} = \frac{1}{2\pi\sigma_Z\sigma_R\sqrt{1-\rho^2}} e^{\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(z-\mu_Z)^2}{\sigma_Z^2} + \frac{(z-\mu_R)^2}{\sigma_R^2} - \frac{2\rho(Z-\mu_Z)(R-\mu_R)}{\sigma_Z\sigma_R}\right)\right)}. \tag{17}$$

As z and R are relevant, we transform our variables into two irrelevant variables (a, b).

We next use the method of substitution to integrate functions.

Set $\mathrm{a} = \mathrm{z}; b = R - \mu_R - \rho\sigma_R z$.

$$\mathrm{f(z,R)} = \mathrm{f(a,b)} = \frac{e^{-\frac{a^2}{2\sigma_a^2}}}{\sqrt{2\pi\sigma_a^2}} \frac{e^{-\frac{b^2}{2\sigma_b^2}}}{\sqrt{2\pi\sigma_b^2}} |\mathrm{J}|\mathrm{dadb} = \frac{e^{-\frac{a^2}{2\sigma_a^2}}}{\sqrt{2\pi\sigma_a^2}} \frac{e^{-\frac{b^2}{2\sigma_b^2}}}{\sqrt{2\pi\sigma_b^2}} \mathrm{dadb}. \tag{18}$$

$$|\mathrm{J}| = \begin{vmatrix} \frac{dz}{da} & \frac{dz}{db} \\ \frac{dR}{da} & \frac{dR}{db} \end{vmatrix} = \begin{vmatrix} 1 & -\frac{1}{\rho\sigma_R} \\ 0 & 1 \end{vmatrix} = 1.$$

Now we need to find the new range of a and b:

$$\mathrm{b} = (\mathrm{R} - M_R) - \rho\sqrt{S_r}a \;\rightarrow R = (b + M_r) + \rho\sqrt{S_r}a. \tag{19}$$

$$H_T = H_t e^{\left(R - \frac{\sigma_h^2\tau}{2}\right) + \sigma\sqrt{\tau}z} < X\, f_\tau \tag{20}$$

$$\mathrm{Z} \leq \frac{\ln\frac{X\, f_\tau}{H_t} - \left(R - \frac{1}{2}\sigma_h^2\tau\right)}{\sigma_h\sqrt{\tau}} = \mathrm{m} - \mathrm{nR}.$$

Here, $\mathrm{m} = \frac{ln\frac{Xf_t}{H_t} + \frac{1}{2}\sigma_h^2\tau}{\sigma\sqrt{\tau}}$, $\mathrm{n} = \frac{1}{\sigma\sqrt{\tau}}$.

$$\mathrm{a} = \mathrm{z} < m - nR = m - n\left(b + M_r + \rho\sqrt{S_r}a\right). \tag{21}$$

$$\mathrm{a} < \frac{(m - nM_r) - nb}{1 + n\rho\sqrt{S_r}} = \mathrm{k} - n_0\mathrm{b}.$$

$$\mathrm{k} = \frac{(m - nM_r)}{1 + n\rho\sqrt{S_r}}, \quad n_0 = \frac{n}{1 + n\rho\sqrt{S_r}}.$$

After finding the reasonable range, we can derive the value of a reverse mortgage as:

$$V_{3,t} = I_1 + I_2. \tag{22}$$

Next, we calculate I_1 and I_2 separately.

$$I_1 = \int_{-\infty}^{\infty} f(b) \int_{-\infty}^{k - n_0 b} H_t e^{-\frac{\sigma_h^2 \tau}{2} + \sigma_h \sqrt{\tau} a} \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \, dadb. \tag{23}$$

$$= H_t \int_{-\infty}^{\infty} f(b) \int_{-\infty}^{k - n_0 b} \frac{e^{-\frac{a^2 - 2\sigma_h\sqrt{\tau} a + \sigma_h^2 \tau}{2}}}{\sqrt{2\pi}} \, dadb. \tag{23-1}$$

$$= H_t \left(a + n_0 b < k \middle| a + n_0 b \sim N\left(\sigma_h \sqrt{\tau} + n_0 \mu_b, 1 + n_0^2 \sigma_b^2\right) \right). \tag{23-2}$$

$$= H_t \mathrm{N} \left(\frac{k - \sigma_h \sqrt{\tau}}{\sqrt{1 + n_0^2 (1 - \rho^2) S_r}} \right). \tag{23-3}$$

$$b = R - \mu_R - \rho \sigma_R a \rightarrow \mathrm{R} = \mathrm{b} + \mu_R + \rho \sigma_R a. \tag{24}$$

$$\mu_b = 0; \; \sigma_{\mathrm{b}}^2 = (1 - \rho^2) S_{r.} \tag{25}$$

$$I_2 = X f_t \int_{-\infty}^{\infty} f(b) \int_{k - n_0 b}^{\infty} e^{-(b + M_r) - \rho\sqrt{S_r} a} f(a) \, dadb. \tag{26}$$

$$= X f_\tau e^{-M_r} \int_{-\infty}^{\infty} \frac{e^{-\frac{b^2 + 2\sigma_b^2 b + \sigma_b^4 - \sigma_b^4}{2\sigma_b^2}}}{\sqrt{2\pi\sigma_b^2}} \int_{k - n_0 b}^{\infty} \frac{e^{-\frac{a^2 + 2\rho\sqrt{S_r} a + (\sqrt{S_r} a)^2 - (\sqrt{S_r} a)^2}{2}}}{\sqrt{2\pi}} \, dadb. \tag{26-1}$$

$$= X f_\tau e^{-M_r + \frac{\sigma_b^2}{2} + \frac{\rho^2 S_r}{2}} P\left(a + n_0 b > k \middle| a + n_0 b \sim N\left(n_0 \sigma_b^2 + \rho\sqrt{S_r}, 1 + n_0^2 (1 - \rho^2) S_r\right) \right). \tag{26-2}$$

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$$= X f_\tau e^{-M_r + \frac{(1-\rho^2) S_r}{2} + \frac{\rho^2 S_r}{2}} N \left(\frac{\rho\sqrt{S_r} - k}{\sqrt{1 + n_0^2 (1 - \rho^2) S_r}} \right). \tag{26-3}$$

$$= X\, f_{\tau} e^{-M_r+\frac{S_r}{2}} N\left(\frac{-k+\rho\sqrt{S_r}}{\sqrt{1+n_0^2(1-\rho^2)S_r}}\right). \tag{26-4}$$

2.4 Uncertain termination time

Under a stochastic termination time assumption, a reverse mortgage can be viewed like the portfolio in section 2.2 with different termination times. Therefore, we use f(t) as the weight in each contract. As a result, the value of a reverse mortgage under the assumption that the house price H_t is governed by a Geometric Brownian motion process and the interest rate is certain can be written as:

$$V_{4,t} = \int_t^{\infty} V_{2,t} f(T|t)dT = E(V_{2,T}|t). \tag{27}$$

An alternate case is that the house price, H_t, is governed by a Geometric Brownianmotion process and the interest rate, r ,follows the Ornstein-Uhlenbeck (OU) process. In that case, the value of a reverse mortgage can be written as:

$$V_{4,t} = \int_t^{\infty} V_{3,t} f(T|t)dT = E\left(V_{3,T}\middle|t\right). \tag{28}$$

Corollary 3: Given that the house price, H_t, is governed by a Geometric Brownian motion process, the interest rate r is deterministic, and the termination time of reverse mortgage T is a random variable, if we set $\tau = \mathrm{T} - \mathrm{t}$, then the value of a reverse mortgage is:

$$V_{4.t} = V_{2,t}(\tau^*) + \frac{V_{2,t}^{*}(\tau^*)}{2} var(\tau). \tag{29}$$

Here, $\tau^* = E(\tau)$.

Proof:

Since $V_{2,t}(\tau) \approx V_{2,t}(\tau^*) + V_{2,t}'(\tau^*)(\tau-\tau^*) + \frac{V_{2,t}^{*}(\tau^*)}{2}(\tau-\tau^*)^2$, and we set $\tau^* = E(\tau)$, we can derive $V_{4.t}$ by using Taylor expansion:

$$
\begin{aligned}
V_{4.t} &= \int_0^\infty V_{2,t}(\tau) f(\tau) d\tau. \\
&= \int_0^\infty \left[V_{2,t}(\tau^*) + V_{2,t}'(\tau^*)(\tau - \tau^*) + \frac{V_{2,t}''(\tau^*)}{2} (\tau - \tau^*)^2 \right] f(\tau) d\tau. \\
&= V_{2,t}(\tau^*) + V_{2,t}'(\tau^*) E(\tau - \tau^*) + \frac{V_{2,t}''(\tau^*)}{2} E(\tau - \tau^*)^2. \\
&= V_{2,t}(\tau^*) + \frac{V_{2,t}''(\tau^*)}{2} var(\tau).
\end{aligned}
$$

Corollary 4: Given that the house price, H_t, is governed by a Geometric Brownian motion process, the interest rate r follows the Ornstein-Uhlenbeck (OU) process, the termination of reverse mortgage T is a random variable, and we set $\tau = T - t$, the reverse mortgage value is:

$$V_{4.t} = V_{3,t}(\tau^*) + \frac{V_{3,t}^*(\tau^*)}{2} var(\tau). \tag{30}$$

Here, $\tau^* = E(\tau)$.

Proof:

$$
\begin{aligned}
V_{4.t} &= \int_0^\infty V_{3,t}(\tau) f(\tau) d\tau. \\
&\approx \int_0^\infty \left[V_{3,t}(\tau^*) + V_{3,t}'(\tau^*)(\tau - \tau^*) + \frac{V_{3,t}''(\tau^*)}{2} (\tau - \tau^*)^2 \right] f(\tau) d\tau. \\
&= V_{3,t}(\tau^*) + V_{3,t}'(\tau^*) E(\tau - \tau^*) + \frac{V_{3,t}''(\tau^*)}{2} E(\tau - \tau^*)^2. \\
&= V_{3,t}(\tau^*) + \frac{V_{3,t}^*(\tau^*)}{2} var(\tau).
\end{aligned}
$$

Corollary 3 is a special case of Corollary 4 when we set the volatility of the interest rate equal to zero and set the mean reversion speed, α, equal to r.

The approximate pricing formula only requires an expectation of the termination time and its variance. Under Corollaries 3 and 4, the termination time of reverse mortgage T only requires being a random variable of the probability distribution it follows. To examine the robustness of our model, we use two commonly used probabilistic models to examine the sensitivity of our approximation formula's results to the estimates of termination time.

2.5 Longevity risk

Since senior homeowners can borrow and stay in their home until they die, or move out of the home permanently, mortgage termination is uncertain. Because most participants remain in their homes until death, the longevity risk is one of the most important risk factors for reverse mortgage holders. Longevity risk occurs, because increasing life expectancy trends among reverse mortgage holders can result in payout levels that are higher than initially expected. In order to measure longevity risk, researchers usually apply probabilistic models to predict the uncertain termination. Although there are many probabilistic models that can be used to describe the evolution of termination, we employ the exponential distribution and life table to capture termination.

2.6 Exponential distribution

If the termination time of the reverse mortgage T follows an exponential distribution, then the instantaneous rate of termination is assumed to be constant. We assume that there is a constant rate parameter λ in the exponential distribution that lasts for the life of the reverse mortgage. The probability density function of an exponential distribution is:

$$f(T) = \lambda e^{-\lambda T}; T \geq 0, \tag{31}$$

and the cumulative distribution function is given by:

$$F(T) = 1 - e^{-\lambda T}; T \geq 0. \tag{32}$$

In this case, the termination expectancy is:

$$E[T] = \int_0^\infty T\lambda e^{-\lambda T} dT = \frac{1}{\lambda}. \tag{33}$$

Termination is constant, because a person always has a mean life expectancy of $\frac{1}{\lambda}$. In reality, life expectancy would decrease and so we adopt a "term structure" of life expectancies in order to build a realistic model.[3]

The memoryless of an exponential distribution is not realistic for the reason that its fourth rank moment is 85.5 times larger than the life table, while the third rank moment is over a hundred times larger. Therefore, we modify life expectancy by truncating the exponential distribution to T<a, which modifies the probability distribution function to $\mathrm{f(T)} = \frac{\lambda e^{-\lambda T}}{1-e^{-\lambda a}}$; $0 \leqq \mathrm{T} \leqq \mathrm{a}$, while the cumulative probability distribution function changes to $\mathrm{F(T)} = \int_0^T \frac{\lambda e^{-\lambda T}}{1-e^{-\lambda a}} \mathrm{dx} = \frac{1-e^{-\lambda T}}{1-e^{-\lambda a}}$; $0 \leqq \mathrm{T} \leqq \mathrm{a}$.

2.7 Life table

A life table (also called a mortality table or actuarial table) provides the rate of deaths occurring in a defined population during a selected time interval. It provides the probability of a person's death before the next birthday, based on current age. Because it is usually used to calculate the remaining life expectancy of people of different ages, researchers use life tables to predict mortgage termination times, which is intuitively appealing for use in reverse mortgages as well. Since the purpose of a reverse mortgage is to supplement retirement income to increase the quality of life of a borrower and as Boehm and Ehrhardt (1994) and Klein and Sirmans (1994) report that the typical borrower is 75 years of age, it seems reasonable that the expected termination time would roughly coincide with death. Our life table data come from The Official Website of the U.S. Social Security Administration.

If a homeowner is *x* years old, then we can use a life table to get the probability of the homeowner's death for each year in the future. We can also assume that the mortgage loan will be repaid through the sale of the house at the end of the year after the death of the homeowner.

If we let T be the time when the house is sold, and use $\mathrm{P(T = k) = P(k -}$

[3] The memoryless property of an exponential property means a newborn's future lifetime has the same distribution as that of a 75-year-old person. However, the reason we use an exponential distribution is that higher-order moments are larger, which could make our similar formula less accurate and even inapplicable. Therefore, the real residual life distribution on higher-order moments should be smaller than the exponential distribution. Thus, it can serve as a test of robustness of our model.

$1 < T \leq k)$, for $\mathrm{T} = 0,1,2\ldots\mathrm{N}$, to denote the probability of death, where $\mathrm{x} + \mathrm{N}$ is the maximum age, then this implies that the homeowner can live for $\mathrm{k} - 1$ more months, but does not survive in the k^{th} month. $\mathrm{P(T = k)}$ denotes the probability of death occurring between ages$\mathrm{x} + \mathrm{k} - 1$ and $\mathrm{x} + \mathrm{k}$.

To obtain $\mathrm{P(T = k)}$ from the life table, we let $l_x$be the number of people who surviveuntil age x. A life table usually begins with 100,000, and so $l_0 = 100{,}000$ and $l_0 > l_1 > l_2 > \cdots > l_n$. If all the people were expected to die before the age of 100 in the life table, thenl_{100} would equal zero. Here, $l_{x+k-1} - l_{x+k}$ is the number of people dying during the year $x + k$, conditional on the homeowner surviving to age *x*. Therefore, $\mathrm{P(T = k)}$ is given by:

$$\mathrm{P(t = k)} = \frac{\mathrm{I_{x+k+1} - I_{x+k}}}{\mathrm{I_x}} \quad \text{for t=0,1,2…(100-x).} \tag{34}$$

3. Comparative analysis

In order to compute the value of a reverse mortgage portfolio, a Monte Carlo simulation could be used. However, a simulation can be computationally burdensome and time-consuming. Therefore, when an issuer of reverse mortgages needs to analyze millions of loans in its portfolio, it is not likely to utilize Monte Carlo simulation to analyze individual loans.

Using our approximate pricing formula, the value of a reverse mortgage portfolio can be easily computed if the expectation and variance of the termination time are known. Using the mean (and variance) of the life expectancy of people at various ages as proxies for the termination time, the value of reverse mortgage portfolio can be computed with reasonable accuracy. If the termination time of the reverse mortgage follows an exponential distribution (μ_i), then μ_i is the expected value of the distribution for age i, w_i is the number of reverse mortgage loans under that age grouping, and the value of the reverse mortgage portfolio can be expressed as:

$$\sum_{i=1}^{n} w_i [V(\mu_i) + \frac{V^{"}(\mu_i)}{2} \mu_i^2]. \tag{35}$$

A reverse mortgage portfolio may be comprised of a number of individual loans with various life expectancies. If the distribution of the termination time follows an exponential distribution, then any previous analysis of comparing the approximation formula results to the simulation results will reveal that the approximation formula may result in a small over- or undervaluation(the valuation errors can be positive or negative). However, the errors can offset each other when mortgages are combined into a portfolio that reduces the magnitude of the pricing error.

For reverse mortgage issuers, this approximate solution can help them quickly calculate the profit and loss balance of r_c, because the amount B_0 lent by the issuer and the approximate formula excluding r_c are both exogenous variables. Therefore, if all other parameters are known, then the computer can quickly calculate the breakeven contract rate r_c. In addition, when $\mathrm{var}(\tau)$ is known, $V_{3,t}(\tau^*)$ can be taken as a function of otherparameters shown in equations (36) and (37).

$$V_{4,t} = V_{3,t}(\tau^*) + \frac{V_{3,t}(\tau^*)}{2}\mathrm{var}(\tau)\,. \tag{36}$$

$$\frac{\partial V_{4,t}}{\partial \theta} = \frac{\partial V_{3,t}(\tau^*)}{\partial \theta} + \frac{1}{2}\frac{\partial V_{3,t}(\tau^{''})}{\partial \theta}\mathrm{var}(\tau);\ \theta = r_0, \sigma_r, \mu_r, \sigma_h, \ldots \ldots, \tag{37}$$

The reverse mortgage issuer can apply numerical methods in order to quickly measure the impact of parameter changes on the lending value of $V_{3,t}$or $V_{4,t}$ and develop hedging strategies thereafter. For example, given $\frac{\partial V_{4,t}}{\partial r_0} = k$, we are able to find a bond with a modified duration of *k* to use as a hedging tool.

We use the life table and the exponential distributions to confirm the difference between the results of the approximation formula. With the homeowner's death at t and the required rate of the bank, the unpaid balance of the lending amount from bank is:

$$F_t = x(e^{rt} - 1)/(1 - e^{-r\Delta})\,. \tag{38}$$

We assume that the probability of the death follows a truncated exponential distribution. Life expectancy is altered to 18.94 years. Under this distribution, we see that:

$$\mathrm{E(T)} = \int_0^a \frac{T\lambda e^{-\lambda T}}{1-e^{-\lambda a}} dT = \frac{1}{1-e^{-\lambda a}}\left\{-Te^{-\lambda T}\Big|_0^a - \frac{1}{\lambda}e^{-\lambda T}\Big|_0^a\right\} = \frac{-ae^{-\lambda a}}{1-e^{-\lambda a}} + \frac{1}{\lambda} = 18.94.$$

We finally conduct the simulation by using the truncated exponential distribution. In this research, "a" is 50, which means the maximum life expectancy is 115 years old.[4] Although truncated to 115, the fourth rank moment is still 5.56 times larger than the life table, and the absolute value of the third rank moment is still 14.4 times larger.

	M1	M2	M3	M4
Life table	18.94	78.84601	-91.6316	13538.66
Truncated exponential	18.94	185.9273	1321.769	75154.3
Exponential	18.94	358.7236	13588.45	1158144

The truncated exponential distribution simulation and approximation value can be seen in Table 1 to Table 5. Because considering a reverse mortgage with LTV greater than 0.5 is unreasonable, we modify the maximum LTV of the life table to be lower than 0.5.

If the discount rate is r_c, then the expected present value of the bank loan is:

$$\int^{\infty} Xe^{-r_c\tau} f_\tau g(\tau)d\tau \,\square. \tag{39}$$

Here, g is the probability density function of the loan's remaining life. The expected present value of the loan= from the bank should equal LTV times H_0:

$$\int^{\infty} Xe^{-r_c\tau} f_\tau g(\tau)d\tau = E(Xf_\tau) = LTV * H_0 \,\square \tag{40-1}$$

$$\lambda = \frac{1}{18.93}\ ; r = 4\%\ ; LTV = 0.4\ ; H = 5000000. \tag{40-2}$$

$$\mathrm{x} = \frac{2000000}{E(e^{-r_c\tau} f_\tau)} = 13299.13336 \tag{40-3}$$

[4] The oldest deceased person for the last decade in the U.S. was 116 years old.

We assume the constant rate parameter of exponential distribution is $\lambda = 1/18.93$, and the homeowner's age x equals 65.[5] For the approximate pricing formula, we utilize the parameters listed below.

Simulation parameter setting table

H= Housing price	5000000
λ = Exponential parameter (Lamda)	1/18.93
σ_h= Volatility of housing Price[6]	0.05
r_f= Risk-free rate	0.02
μ_{-} =Long-term mean reverting level	0.04
σ_r =Volatility of interest rate	0.08
α =Mean-reverting speed	0.25
ρ=Correlation between H and r	0.25

$\mu_r = 0.04, \sigma_r = 0.08, \alpha = 0.25, \rho = 0.25$[7]

Table 1 and Figure 1 illustrate the relationship between the simulations and approximations of the exponential distribution and the life table, while sigma R varies from 0.05% to 1.25%. Table 1 shows that the error of the life table increases as sigma R increases. On the other hand, the exponential distribution has a negative error at 0.05%, 0.10%, and 0.25%, while fluctuating all the way up to 1.25%. Both the exponential distribution and the life table have increasing errors while sigma R rises. Figure 1 reveals that the life table has a constant and mild increase in error from 0% to 2%, while the exponential distribution

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5 Per the life table adopted, the average life expectancy of the elderly aged over 65 in Taiwan is about to be 65 + 18.94. The average life expectancy of the three distributions is set at 18.94 since average life expectancy of the exponential distribution is 18.94. The average of 18.93 might be a small error caused by rounding probability. Since the approximate formula of our research is distribution free, the estimated average and standard deviation of any countries' future life expectancy can be applicable.

6 The parameters are taken from Buist and Yang (1998), Yang, Lin and Cho (2011), and Stephen *et al*. (1995). The σ_h value used in Stephen *et al*. (1995) is 0.05, 0.1 in Buist and Yang (1998), and 0.02 in Yang, Lin and Cho (2011). Therefore, we employ the median of these for our simulation parameter.

7 Regarding μ_r, we use the 30-year U.S. treasury rate in 2010. Chan *et al*. (1992), Buist and Yang (1998), Lin *et al*. (2006), and Yang, Lin and Cho (2011) provide values for σ_r of 0.08, 0.03, 0.15, and 0.15, respectively. We choose 0.08 as the volatility measure. However, because those measures are based on a CIR model, we need to convert them for the Vasicek (1977) model that we use, which results in a value of 0.01 for our simulation parameter.

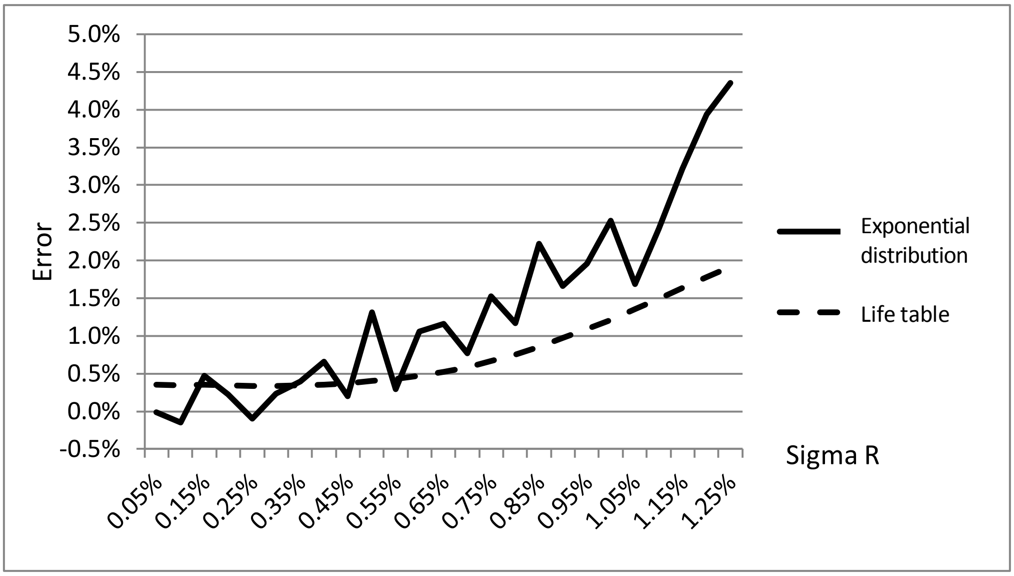

Figure 1
The relationship between interest rate volatility and reverse mortgage value

Table 1
Relationship between interest rate volatility and reverse mortgage value

	Exponential distribution				Life table			
Sig_r	**Simulation**	**Approximation**	**Error**	**Error%**	**Simulation**	**Approximation**	**Error**	**Error%**
0.05%	2015101.347	2014873.055	-228.2923846	-0.01%	2158271.07	2165897.748	7626.678174	0.35%
0.10%	2018560.622	2015498.552	-3062.070112	-0.15%	2158846.329	2166269.975	7423.646628	0.34%
0.1500%	2007683.097	2017117.535	9434.437278	0.47%	2159673.269	2167271.603	7598.333724	0.35%
0.2000%	2014540.508	2019080.206	4539.697418	0.23%	2160760.023	2168215.9	7455.876758	0.35%
0.2500%	2022605.975	2020737.103	-1868.872314	-0.09%	2162115.12	2169472.164	7357.043645	0.34%
0.3000%	2018465.97	2023283.987	4818.017231	0.24%	2163747.289	2171088.961	7341.672507	0.34%
0.3500%	2017896.478	2025963.693	8067.215654	0.40%	2165665.269	2173161.365	7496.09591	0.35%
0.4000%	2016392.616	2029647.34	13254.72358	0.66%	2167877.627	2175601.553	7723.925951	0.36%
0.4500%	2029781.303	2033904.332	4123.028575	0.20%	2170392.592	2178414.25	8021.657857	0.37%
0.5000%	2012313.638	2038738.715	26425.07721	1.31%	2173217.91	2181926.262	8708.351323	0.40%
0.5500%	2037845.779	2043829.566	5983.787365	0.29%	2176360.717	2185730.95	9370.233481	0.43%
0.6000%	2028887.612	2050375.837	21488.22506	1.06%	2179827.436	2190110.996	10283.56008	0.47%
0.6500%	2033390.311	2056972.597	23582.28574	1.16%	2183623.703	2195074.626	11450.92264	0.52%
0.7000%	2048774.662	2064602.929	15828.26691	0.77%	2187754.304	2200539.174	12784.86929	0.58%
0.7500%	2041951.721	2073165.25	31213.52977	1.53%	2192223.149	2206835.888	14612.73925	0.67%
0.8000%	2058188.956	2082232.9	24043.94426	1.17%	2197033.252	2213654.293	16621.04116	0.76%
0.8500%	2046305.045	2091705.181	45400.13574	2.22%	2202186.739	2221005.882	18819.14266	0.85%
0.9000%	2068071.331	2102458.393	34387.06177	1.66%	2207684.871	2229132.221	21447.34974	0.97%
0.9500%	2072472.753	2113090.136	40617.38283	1.96%	2213528.071	2237724.608	24196.53694	1.09%
1.0000%	2072141.092	2124477.037	52335.94444	2.53%	2219715.976	2246612.348	26896.37233	1.21%
1.0500%	2100825.331	2136301.998	35476.66675	1.69%	2226247.484	2256405.112	30157.62883	1.35%
1.1000%	2097352.218	2148030.758	50678.53947	2.42%	2233120.817	2266427.529	33306.71255	1.49%
1.1500%	2093048.509	2160322.538	67274.02832	3.21%	2240333.582	2276967.843	36634.26142	1.64%
1.2000%	2090174.181	2172533.972	82359.7914	3.94%	2247882.838	2287855.029	39972.19169	1.78%
1.2500%	2094132.765	2185214.964	91082.19823	4.35%	2255765.16	2299055.266	43290.10632	1.92%

fluctuates from -8% to 8%. Another trend worth pointing out is that the exponential distribution error increases violently between 1.05% and 1.25%.

Table 2 and Figure 2 illustrate the relationship between the simulations and approximations of the exponential distribution and the life table. The error percentage of the life table is roughly stable at 0.2% when LTV ranges from 0.02 to 0.34. After that, it slightly increases until 1%. The exponential distribution error percentage generally has a violent change every 4 ticks. Additionally, we also observe that the error gradually increases when LTV is larger than 0.2 up until 0.46.

Table 3 and Figure 3 illustrate the relationship between the simulations and approximations of the exponential distribution and the life table, while Sigma H varies from 0.5% to 10%. According to Table 3, the errors of the exponential distribution and the life table reach the maximum (respectively) when Sigma H is 5.5% and 6.0%. Figure 3 reveals that the life table has almost perfect correspondence among the simulations and approximations with errors stably fluctuating between -1% and 1%, whereas the exponential distribution has intensive fluctuations. After increasing along with Sigma H from 0.5% to 10%, the error percentages drop noticeably all the way down until -4.13% with a range of more than 6%. Another point we would like to emphasize is that the error is rather small under a reasonable price (mortgage value above US$2 million). If Sigma H is large, then having a minor risk premium would cause the mortgage value to be lower than the present value of US$2 million and therefore incur a larger error.

Table 4 and Figure 4 illustrate the relationship between the simulations and approximations of the exponential distribution and the life table, while the risk premium varies from 0.002 to 0.04. Figure 4revealsthat the life table has a constantly increasing error from 0% to 4%. As for the exponential distribution, the error percentages skyrocket from 2% to 4.5% and suddenly drop violently at 0.3 percent. Therefore, we can infer that 0.3 might be a critical point for the risk premium.

Table 5 and Figure 5 illustrate the relationship between the simulations and approximations of the exponential distribution and the life table, while lo varies from -0.3 to 0.3.Figure 5 reveals that the life table has a constant increase in error from 0% to 1%, while the exponential distribution's error seesaws from -0.33%

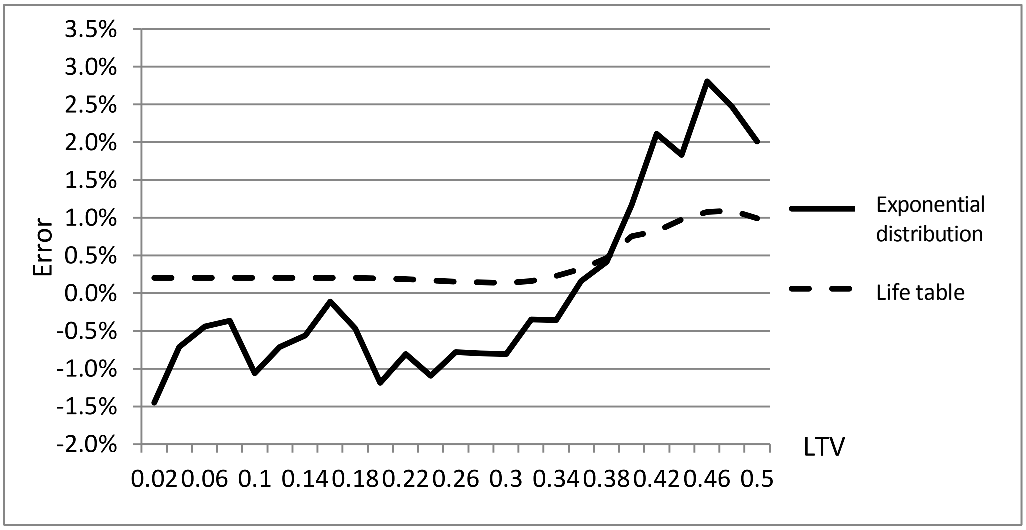

Figure 2
Relationship between loan to value (LTV) and reverse mortgage value

Table 2
Relationship between loan to value (LTV) and reverse mortgage value

	Truncated exponential distribution				Life table			
LTV	**Simulation**	**Approximation**	**Error**	**Error%**	**Simulation**	**Approximation**	**Error**	**Error%**
0.02	103508.014	102011.5507	-1496.463306	-1.45%	108453.2052	108675.4149	222.2097519	0.20%
0.04	205478.7902	204023.1014	-1455.688812	-0.71%	216906.4104	217350.8299	444.4195037	0.20%
0.06	307375.5228	306014.3046	-1361.218252	-0.44%	325359.6156	326023.3764	663.7608383	0.20%
0.08	409548.6494	408046.2028	-1502.446612	-0.37%	433812.8225	434701.6598	888.8372933	0.20%
0.1	515580.3247	510105.2309	-5475.093805	-1.06%	542266.0691	543379.9431	1113.873988	0.21%
0.12	616464.7537	612055.7391	-4409.014585	-0.72%	650719.6997	652046.7529	1327.053178	0.20%
0.14	718105.7114	714114.7673	-3990.944065	-0.56%	759175.3856	760759.457	1584.071488	0.21%
0.16	816950.6165	816065.2758	-885.3406842	-0.11%	867638.3763	869414.7934	1776.417106	0.20%
0.18	922451.3237	918151.4379	-4299.885802	-0.47%	976120.6582	978104.5528	1983.894634	0.20%
0.2	1032424.828	1020210.501	-12214.32669	-1.18%	1084643.666	1086782.859	2139.192736	0.20%
0.22	1131179.314	1122052.704	-9126.609694	-0.81%	1193238.809	1195461.289	2222.479798	0.19%
0.24	1237741.885	1224166.994	-13574.89104	-1.10%	1301944.597	1304117.326	2172.729538	0.17%
0.26	1337032.964	1326555.303	-10477.66117	-0.78%	1410800.224	1413004.802	2204.578744	0.16%
0.28	1441206.109	1429710.973	-11495.13694	-0.80%	1519836.465	1522012.742	2176.277226	0.14%
0.3	1546374.368	1533861.937	-12512.43023	-0.81%	1629065.37	1631310.192	2244.822007	0.14%
0.32	1645080.338	1639354.155	-5726.182948	-0.35%	1738470.384	1741303.463	2833.078922	0.16%
0.34	1753509.994	1747251.952	-6258.042183	-0.36%	1847998.271	1852203.441	4205.170328	0.23%
0.36	1854178.906	1857115.238	2936.331812	0.16%	1957553.809	1963913.178	6359.368751	0.32%
0.38	1960990.34	1969114.235	8123.894291	0.41%	2066997.692	2076670.285	9672.592782	0.47%
0.4	2058188.956	2082232.9	24043.94426	1.17%	2197033.252	2213654.293	16621.04116	0.76%
0.42	2147936.112	2193389.754	45453.64253	2.12%	2284782.389	2303585.254	18802.86494	0.82%
0.44	2259332.094	2300786.619	41454.52511	1.83%	2392647.936	2416071.137	23423.20164	0.98%
0.46	2336054.967	2401611.674	65556.7074	2.81%	2499465.464	2526484.8	27019.33631	1.08%
0.48	2431104.702	2491153.026	60048.32364	2.47%	2604940.152	2633432.162	28492.00936	1.09%
0.5	2517876.102	2568534.363	50658.26038	2.01%	2708770.282	2735781.135	27010.85244	1.00%

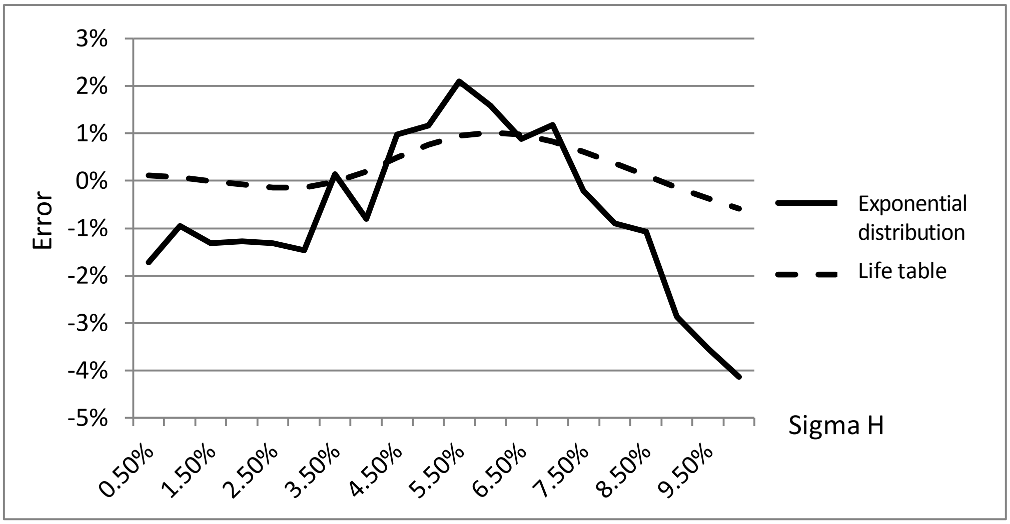

Figure 3
Relationship between the housing price volatility and reverse mortgage value

Table 3
Relationship between the housing price volatility and reverse mortgage value

	Exponential distribution				Life table			
Sig_h	**Simulation**	**Approximation**	**Error**	**Error%**	**Simulation**	**Approximation**	**Error**	**Error%**
0.50%	2076109.065	2040529.444	-35579.62125	-1.71%	2182479.745	2185030.298	2550.553116	0.12%
1%	2060049.849	2040420.935	-19628.91357	-0.95%	2183482.686	2185076.201	1593.514705	0.07%
1.50%	2067709.865	2040529.625	-27180.23978	-1.31%	2184976.131	2184892.764	-83.36658104	0.00%
2%	2067075.24	2040748.382	-26326.85804	-1.27%	2186934.411	2185169.713	-1764.698475	-0.08%
2.50%	2069838.732	2042712.632	-27126.09998	-1.31%	2189216.943	2186052.707	-3164.235954	-0.14%
3%	2077858.913	2047424.158	-30434.75491	-1.46%	2191595.548	2188488.524	-3107.024077	-0.14%
3.50%	2052981.439	2055802.76	2821.32154	0.14%	2193804.357	2193276.857	-527.4996728	-0.02%
4%	2083543.454	2066722.584	-16820.86939	-0.81%	2195585.574	2199963.486	4377.911227	0.20%
4.50%	2056460.865	2076436.993	19976.12808	0.97%	2196718.729	2207492.472	10773.74328	0.49%
5%	2058188.956	2082232.9	24043.94426	1.17%	2197033.252	2213654.293	16621.04116	0.76%
5.50%	2038360.146	2081111.443	42751.29775	2.10%	2196409.379	2217167.881	20758.50216	0.95%
6%	2040385.677	2072724.601	32338.9245	1.58%	2194772.65	2217176.269	22403.61973	1.02%
6.50%	2039103.771	2057118.301	18014.53036	0.88%	2192085.853	2213452.919	21367.06571	0.97%
7%	2012488.973	2036277.693	23788.71945	1.18%	2188340.695	2206478.528	18137.83308	0.83%
7.50%	2015536.535	2011436.11	-4100.425567	-0.20%	2183550.307	2197055.987	13505.68004	0.62%
8%	2002255.019	1984174.959	-18080.06026	-0.90%	2177743.021	2185733.644	7990.622566	0.37%
8.50%	1977310.44	1955997.178	-21313.26154	-1.08%	2170957.442	2173321.272	2363.829904	0.11%
9%	1985518.71	1928546.276	-56972.4344	-2.87%	2163238.678	2160113.882	-3124.795474	-0.14%
9.50%	1971274.15	1901434.329	-69839.82155	-3.54%	2154635.541	2146533.441	-8102.100248	-0.38%
10%	1956394.675	1875602.516	-80792.15857	-4.13%	2145198.516	2132666.348	-12532.16818	-0.58%

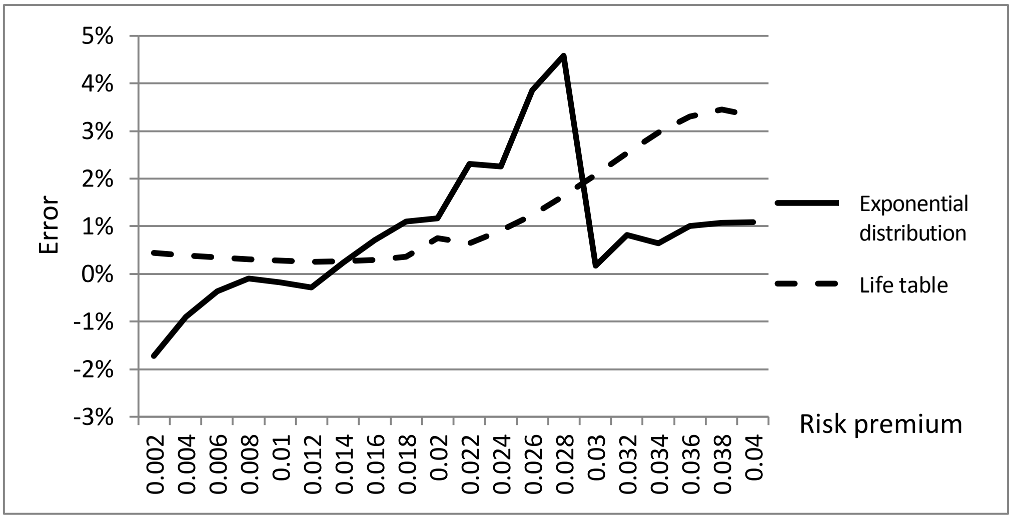

Figure 4
Relationship between risk premiums and reverse mortgage value

Table 4
Relationship between risk premiums and reverse mortgage value

	Exponential distribution				Life table			
Risk premium	**Simulation**	**Approximation**	**Error**	**Error%**	**Simulation**	**Approximation**	**Error**	**Error%**
0.002	1590253.923	1562788.325	-27465.59826	-1.73%	1412618.131	1418776.769	6158.638325	0.44%
0.004	1623532.406	1608963.782	-14568.62427	-0.90%	1472212.839	1477997.61	5784.77124	0.39%
0.006	1663620.436	1657631.42	-5989.015853	-0.36%	1534654.296	1540084.883	5430.586182	0.35%
0.008	1710399.322	1708663.69	-1735.631843	-0.10%	1600126.274	1605063.002	4936.727732	0.31%
0.01	1765971.414	1762965.686	-3005.728364	-0.17%	1668825.286	1673490.037	4664.750347	0.28%
0.012	1825130.52	1820033.56	-5096.959885	-0.28%	1740955.488	1745472.27	4516.781464	0.26%
0.014	1876093.601	1880613.351	4519.75018	0.24%	1816719.685	1821537.876	4818.191172	0.27%
0.016	1930672.768	1944313.624	13640.8561	0.71%	1896305.608	1901812.91	5507.302	0.29%
0.018	1989734.279	2011504.592	21770.31286	1.09%	1979867.026	1987010.474	7143.447824	0.36%
0.02	2058188.956	2082232.9	24043.94426	1.17%	2197033.252	2213654.293	16621.04116	0.76%
0.022	2107744.859	2156438.556	48693.6971	2.31%	2159214.866	2173180.429	13965.56255	0.65%
0.024	2185464.245	2234714.363	49250.11749	2.25%	2254907.838	2275392.631	20484.79245	0.91%
0.026	2229881.26	2315918.13	86036.86983	3.86%	2354332.7	2383448.348	29115.6482	1.24%
0.028	2294815.443	2399993.674	105178.2311	4.58%	2457079.376	2497303.93	40224.5536	1.64%
0.03	2054207.833	2057882.432	3674.599737	0.18%	2562561.982	2615825.016	53263.03428	2.08%
0.032	2043782.816	2060587.681	16804.86542	0.82%	2670020.661	2737921.535	67900.87433	2.54%
0.034	2050121.492	2063412.09	13290.59815	0.65%	2778539.611	2860922.095	82382.48379	2.96%
0.036	2045577.168	2066030.537	20453.36862	1.00%	2887081.781	2982612.07	95530.28906	3.31%
0.038	2047167.618	2069094.559	21926.94121	1.07%	2994538.295	3097939.935	103401.6398	3.45%
0.04	2050128.272	2072278.996	22150.72367	1.08%	3099788.282	3202463.106	102674.8244	3.31%

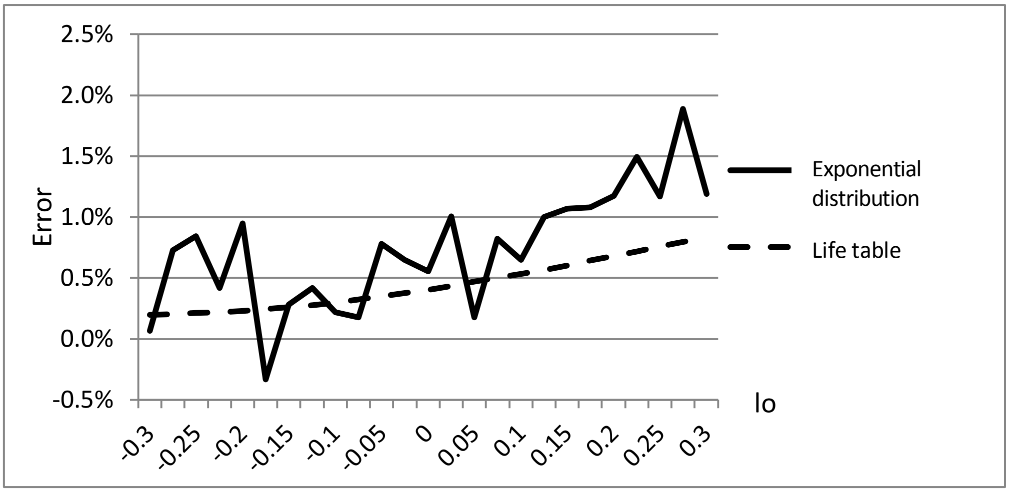

Figure 5
Relationship between lo (ρ) and reverse mortgage value

Table 5
Relationship between lo (ρ) and reverse mortgage value

	Exponential distribution				Life table			
lo	**Simulation**	**Approximation**	**Error**	**Error%**	**Simulation**	**Approximation**	**Error**	**Error%**
-0.3	2035178.621	2036579.637	1401.016916	0.07%	2183097.856	2187396.527	4298.670949	0.20%
-0.275	2022412.214	2037127.949	14715.73453	0.73%	2183613.437	2188015.642	4402.204709	0.20%
-0.25	2020819.909	2037896.571	17076.66168	0.85%	2184146.203	2188776.869	4630.665716	0.21%
-0.225	2030157.206	2038668.991	8511.785229	0.42%	2184695.009	2189451.182	4756.172176	0.22%
-0.2	2020534.626	2039662.79	19128.16376	0.95%	2185258.776	2190222.603	4963.827413	0.23%
-0.175	2047515.204	2040661.476	-6853.728051	-0.33%	2185836.479	2191137.465	5300.985818	0.24%
-0.15	2035939.156	2041665.606	5726.450575	0.28%	2186427.153	2192104.406	5677.252279	0.26%
-0.125	2034685.277	2043218.337	8533.059986	0.42%	2187029.887	2193032.052	6002.165294	0.27%
-0.1	2040324.586	2044777.625	4453.039304	0.22%	2187643.818	2194104.382	6460.563718	0.30%
-0.075	2043069.464	2046669.582	3600.117872	0.18%	2188268.134	2195321.778	7053.643165	0.32%
-0.05	2032788.374	2048677.71	15889.33611	0.78%	2188902.07	2196501.026	7598.956876	0.35%
-0.025	2037239.255	2050476.985	13237.73052	0.65%	2189544.9	2197780.156	8235.255418	0.38%
0	2041627.893	2052936.088	11308.19512	0.55%	2190195.945	2199021.805	8825.860591	0.40%
0.025	2034724.458	2055187.366	20462.90884	1.01%	2190854.56	2200318.065	9463.505525	0.43%
0.05	2054207.833	2057882.432	3674.599737	0.18%	2191520.139	2201806.898	10286.75867	0.47%
0.075	2043782.816	2060587.681	16804.86542	0.82%	2192192.111	2203167.298	10975.187	0.50%
0.1	2050121.492	2063412.09	13290.59815	0.65%	2192869.935	2204583.076	11713.14064	0.53%
0.125	2045577.168	2066030.537	20453.36862	1.00%	2193553.104	2205962.654	12409.54973	0.57%
0.15	2047167.618	2069094.559	21926.94121	1.07%	2194241.138	2207489.797	13248.65945	0.60%
0.175	2050128.272	2072278.996	22150.72367	1.08%	2194933.584	2209072.883	14139.29905	0.64%
0.2	2051478.129	2075584.224	24106.09508	1.18%	2195630.017	2210620.267	14990.25039	0.68%
0.225	2048126.59	2078685.04	30558.4498	1.49%	2196330.032	2212040.282	15710.25007	0.72%
0.25	2058188.956	2082232.9	24043.94426	1.17%	2197033.252	2213654.293	16621.04116	0.76%
0.275	2046895.537	2085577	38681.46311	1.89%	2197739.317	2215232.909	17493.59252	0.80%
0.3	2064593.9	2089151.713	24557.81243	1.19%	2198447.89	2216776.196	18328.30592	0.83%

to 1.89%. One interesting thing from Figure 5 is that in contrast with a beautiful positive trend of the life table, the exponential distribution seems to be more active and also has a positive trend. Moreover, if we cut the figure in half when lo equals 0, then we actually can find out that the chart is similarly upside down, whereby both have a peak respectively at -0.33% and 1.89% and stably fluctuate in an absolute range of 1%.

4. Conclusion

Properly valuing reverse mortgages is an important topic, because these financial instruments are becoming increasingly common and can be an attractive cash tool for senior homeowners to use in order to improve their retirement lifestyle. Thus, the importance of valuing this significant financial instrument is growing as well.

In order to simplify the valuation of these instruments, we provide an approximation formula for valuing an annuity reverse mortgage when the housing price and interest rate are stochastic. Our approximate pricing formula greatly reduces computational intensity, because it only requires an expectation (mean) and a variance of the termination time.

We compare the results of our approximate pricing formula to the results from a Monte Carlo simulation, where the housing price and interest rate are stochastic. The analysis reveals that the difference between the results of the approximation formula and the simulation is small. To examine the robustness of our model, we use multiple commonly used distributional assumptions and find that our approximation formula is robust with respect to distributional assumptions.

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